Instability of Truncated symmetric powers

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Abstract

Let X be a smooth n-dimensional projective variety over an algebraic closed field k with $char\, k > 0$, $F_r : X \to X^1$ be the relative Frobenius morphism and E a vector bundle on X. X.Sun has proved that the instability of $F_{r*}E$ is bounded by the instability of $E \otimes T^l(\Omega_X)(0 \leq l \leq n(p-1))$. It is also known that there is an estimate about the instability of tensor products(and symmetric powers etc.). In this paper we are interested in estimating the instability of the truncated symmetric powers $T^l(E)$ (Theorem 5.6).

1 Introduction

Let X be a smooth n-dimensional projective variety over an algebraic closed field k with $char\, k > 0$, $F : X \to X$ be absolute Frobenius morphisms of X, $F_r : X \to X^1$ be the relative Frobenius morphism and E a vector bundle on X. Mehta and Pauly have showed that if E is semistable then $F_{r*}E$ is also semistable when X is a curve of genus $g \geq 2$ in ([6]). In the higher-dimensional case Sun has proved that the instability of $F_{r*}E$ is bounded by the instabilities of $E \otimes T^l(\Omega_X)(0 \leq l \leq n(p-1))$ in [8]. There are also some further refined discussion in algebraic surface case in [3][9].

It is also known that there is an estimation about the instability of tensor products (and symmetric powers etc.)[5]. In this paper we are interested in giving an estimation about the truncated symmetric powers $T^l(E)$.

We collect some well known results about Harder-Narasimhan polygon([7]), strongly semistability [4][5] and direct image of torsion free sheaves under Frobenius morphism ([8][10]).

We describe the main idea of the proof: Firstly, As we know that when k_0 is larger enough, the torsion free sheaf $F^{k_0*}E$ has a Harder-Narasimhan filtration such that all quotients are strongly semistable, then we can use it

to construct a flat family $T^l(\mathcal{E})$ whose generic fiber is $T^l(F^{k_0*}E)$ and center fiber is $T^l(\oplus gr^i(F^{k_0*}E))$. Secondly, the instability of $T^l(F^{k_0*}E)$ can be bounded by the instability of $T^l(\oplus gr^i(F^{k_0*}E))$ by the upper semi continuity of Harder-Narasimhan polygon. In fact, the bound of $I(T^l(\oplus gr^i(F^{k_0*}E)))$ is computable, since we can use the direct sum decomposition of the long exact sequence to look for the maximum and the minimum slope strong semistable part of it and the slope is computed as symmetry product case. Finally, we can use it to give a bound of instability of $T^l(E)$.

2 Harder-Narasimhan polygon

Let k be an algebraically closed field of any characteristic. Let X be a smooth n-dimensional projective variety over k with an ample divisor H. If E be a rank r torsion-free sheaf on X then one can define its slope:

$$\mu(E) = \frac{c_1(E)H^{n-1}}{r}$$

Then E is slope H-semistable if for any nonzero subsheaf $E \subset F$ we have $\mu(E) \le \mu(F)$. There exists a unique Harder-Narasimhan filtration:

$$0 = E_0 \subset E_1 \subset ... \subset E_m = E,$$

such that

$$\mu_{max}(E) = \mu(E_1) > \mu(E_2/E_1) > ... > \mu(E_m/E_{m-1}) = \mu_{min}(E)$$

and $E_i/E_{i-1}, (1 \le i \le m)$ are semistable torsion free sheaves.

Define the instability of E: $I(E) = \mu_{max}(E) - \mu_{min}(E)$.

For any torsion free sheaf G we may associate the point $p(G) = (rkG, degG)$ in the coordinate plane. Now, we consider the points $p(E_0), ..., p(E_m)$ and connect them successively by line segments and connecting the last point with the first one. The resulting polygon $HNP(E)$ is called the Harder-Narasimhan polygon of E.

Proposition 2.1. *[7] Let $\mathcal{E}$ be an flat family of torsion-free sheaf on X, parameterized by the scheme S. Let $s, s_0 \in S$ with s_0 a specialization of s. Then $HNP(\mathcal{E}_{s_0}) \ge HNP(\mathcal{E}_s)$(in the partial ordering of convex polygons); i.e, the Harder-Narasimahan Polygon rises under specialization. In particular $I(\mathcal{E}_{s_0}) \ge I(\mathcal{E}_s)$.*

3 Strongly semistable

If $p = char\ k > 0$, let $F : X \to X$ be absolute Frobenius morphisms of X,$F_r : X \to X^1$ be the relative k-linear Frobenius morphism, where $X^1 := X \times_k k$ is the base change of X/k under the Frobenius morphism of $Spec(k)$. The natural morphism $X^1 \to X$ is an isomorphism. We say that E is slope strongly H-semistable if for all $m \geq 0$ the pull back $(F^m)^*E$ are slope $(F^m)^*H$-semistable.

For any torsion free sheaf E, there exists an integer k_0 such that for any $k \geq k_0$ and all quotients in the Harder-Narasimhan filtration of $F^{k*}E$ are strongly semistable.

Let

$$L_{max} = \lim_{k \to \infty} \frac{\mu_{max}(F^{k*}(E))}{p^k}, L_{min} = \lim_{k \to \infty} \frac{\mu_{min}(F^{k*}(E))}{p^k}$$

Clearly, $L_{min}(E) = -L_{max}(E^*)$. By definition, E is strongly semistable if and only if $L_{min}(E) = \mu(E) = L_{max}(E)$.

Let $\alpha(E) = max(L_{max}(E) - \mu_{max}(E), \mu_{min}(E) - L_{min}(E))$. We denote

$$L_X := \begin{cases} \frac{L_{max}(\Omega_X)}{p} & \texttt{if}\ \mu_{max}(\Omega_X) > 0 \\ 0 & \texttt{if}\ \mu_{max}(\Omega_X) \leq 0 \end{cases}$$

This give a uniform bound of the difference of instability after Frobenius pull back.

Corollary 3.1. *[4] Let E be a torsion-free sheaf E of rank r, then*

$$\alpha(E) \leq (r-1)L_X.$$

We also know that for any torsion free sheaves E_i $(1 \leq i \leq l)$,

$$L_{max}(\otimes_{i=1}^{l} E_i) = \sum_{i=1}^{l} L_{max}(E_i)$$

and $\otimes, F_r^*, F, \bigwedge, Sym$ of strongly semistable bundles are also strongly semistable.

Corollary 3.2. *Let $E_i, (1 \leq i \leq l)$ be torsion free sheaves on X, then* $I(\otimes_{i=1}^{l} E_i) \leq \sum_{i=1}^{l} I(E_i) + 2(-l + \sum_{i=1}^{l} rk(E_i))L_X$

Proof. We have $L_{max}(\bigotimes_{i=1}^{l} E_i) = \sum_{i=1}^{l} L_{max}(E_i)$ by 2.3.3 in [4]. Since $L_{min}(E) = -L_{max}(E^\vee)$ so we can get

$$L_{min}(\bigotimes_{i=1}^{l} E_i) = -L_{max}(\bigotimes_{i=1}^{l} E_i)^\vee = -L_{max}(\bigotimes_{i=1}^{l}(E_i)^\vee) = \sum_{i=1}^{l} L_{min}(E_i)$$

Hence,

$$\begin{aligned} I(\bigotimes_{i=1}^{l} E_i) &\leq L_{max}(\bigotimes_{i=1}^{l} E_i) - L_{min}(\bigotimes_{i=1}^{l} E_i) = \sum_{i=1}^{l} L_{max}(E_i) - \sum_{i=1}^{l} L_{min}(E_i) \\ &= \sum_{i=1}^{l}(L_{max}(E_i) - \mu_{max}(E_i) - L_{min}(E_i) + \mu_{min}(E_i)) + \sum_{i=1}^{l}(\mu_{max}(E_i) - \mu_{min}(E_i)) \\ &\leq 2(\sum r_i - l)I(X) + \sum_{i=1}^{l} I(E_i). \end{aligned}$$

□

4 Direct images under Frobenius morphism

Let S_l be the symmetric group of l elements with the action on $V^{\otimes l}$ by $(v_1 \otimes \cdots \otimes v_l) \cdot \sigma = v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_l}$ for $v_i \in V$ and $\sigma \in S_l$. Let $e_1, \cdots, e_n$ be a basis of V, for $k_i \geq 0$ with $k_1 + \cdots + k_n = l$ define

$$v(k_1 \otimes \cdots \otimes k_l) = \sum_{\sigma \in S_l} (e_1{}^{\otimes k_1} \otimes \cdots \otimes e_n{}^{\otimes k_n}) \cdot \sigma.$$

Definition 4.1. *[8] Let $T^l(V) \subset V^{\otimes l}$ be the linear subspace generated by all vectors $v(k_1, ..., k_n)$ for all $k_i \geq 0$ satisfying $k_1 + \cdots + k_n = l$. It is a representation of $GL(V)$. If $\mathcal{V}$ is a vector bundle of rank n, the subbundle $T^l(\mathcal{V}) \subset \mathcal{V}^{\otimes l}$ is defined to be the associated bundle of the frame bundle of $\mathcal{V}^{\otimes l}$(which is a principal $Gl(n)$-bundle)through the representation $T^l(V)$.*

By sending any $e_1^{k_1} e_2^{k_2} ... e_n^{k_n} \in Sym^l(V)$ to $v(k_1, ..., k_n)$, we have

$$Sym^l(V) \to T^l(V) \to 0$$

Which is an isomorphism in char 0. When char $k = p$, $T^l(V)$ is isomorphic to the quotient of $Sym^l(V)$ by the relations $e^p = 0$, $1 \leq i \leq n$. $T^l(V)$ is called "**truncated symmetric powers**"[1].

Proposition 4.2. *[8]Let $l(p) \geq 0$ be the unique integer such that $0 \leq l - l(p)p < p$. Then in the category of $GL(n)$-representations, we have long exact sequence*

$$0 \longrightarrow Sym^{l-l(p)p}(V) \otimes_k \bigwedge^{l(p)}(F_r^*V) \longrightarrow Sym^{l-(l(p)-1)p}(V) \otimes_k \bigwedge^{l(p)-1}(F_r^*V) \longrightarrow ...$$

$$\longrightarrow Sym^{l-p}(V) \otimes_k \bigwedge(F_r^*V) \longrightarrow Sym^l(V) \longrightarrow T^l(V) \longrightarrow 0 \qquad (1)$$

For any vector bundle W on X, there exists a canonical filtration

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset ... \subset V_1 \subset V_0 = V = F_r^*(F_{r*}W)$$

Such that the caonical connection $\nabla : V \to V \otimes \Omega_X^1$ induces injective morphisms $V_l/V_{l+1} \stackrel{\nabla}{\to} (V_{l-1}/V_l) \otimes \Omega_X^1$ and the isomorphisms $V_l/V_{l+1} \cong W \otimes T^l(\Omega_X^1)$. Let

$$I(W,X) = Max\{I(W \otimes T^l(\Omega_X))|0 \leq l \leq n(p-1)\}$$

Using this, Sun has proved that instability of F_*W is bounded by instability of $I(W,X)$:

Corollary 4.3. *[8] Let X be a smooth projective variety of dim(X) = n, whose canonical divisor K_X satisfies $K_X \cdot H^{n-1} \geq 0$. Then*

$$I(F_{r*}W) \leq p^{n-1} rk(W) I(W,X).$$

5 Instability of truncated symmetric powers

Remark 5.1. *Any torsion free sheaf $\mathcal{F}$ defines a rational vector bundle V (vector bundle over a big open set). A rational vector bundle V can be extend to many torsion free sheaves, a canonical one is $j_!V$. In our following discussion, since codimension 2 part doesn't affect our slope, for convenience, we can choose a common big open set, such that all torsion sheaves restrict on it are vector bundles. Thus we can only consider the vector bundle case.*

Lemma 5.2. *In the short exact sequence:*

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$$

Any two of E, F, G are semistable(strongly semistable) with the same slope, then the other is trival(codim(support) ≥ 2) or semistable(strongly semistable) with the same slope.

Proof. Assume E, F are semistable with the same slope. If G is nontrival, let G_{m-1} be last item in the Harder-Narasimhan filtration

$$0 = G_0 \subset G_1 \subset ... \subset G_{m-1} \subset G_m = G.$$

By surjective homomorphism $F \longrightarrow G/G_{m-1}$, we know that

$$\mu(G/G_{k-1}) \geq \mu(F) = \mu(G) \geq \mu(G/G_{k-1})$$

So G is semistable. For any integer k, we consider the following exact sequence

$$0 \longrightarrow F^{k*}E \longrightarrow F^{k*}F \longrightarrow F^{k*}G \longrightarrow 0.$$

We get strongly semistable case by a similar discussion. □

Remark 5.3. *In fact, for the long exact sequence:*

$$0 \longrightarrow H_1 \longrightarrow ... \longrightarrow H_{j-1} \longrightarrow H_j \longrightarrow 0$$

As we have showed above, if H_{j-1} is strongly semistable, $\mu(H_i) = u$ or H_i trival $(0 \leq i \leq j-1)$. Then H_j is trival or strongly semistable with $\mu(H_j) = u$. This is enough for our following application.

Proposition 5.4. *Assume $E = E_1 \oplus ... \oplus E_m$ with E_i $(1 \leq i \leq m)$ are strongly semistable torsion free sheaves and $\mu(E_1) \geq \mu(E_2) \geq \cdots \geq \mu(E_m)$, $r = rk(E)$. If $l \geq r(p-1)$, then $I(T^l E) = 0$. If $l < r(p-1)$, then we have*

$$I(T^l E) \leq Min\{l, [r/2](p-1)\} I(E).$$

Proof. Step 1: Construction of Harder-Narasimhan filtration. Since $\otimes, F_r^*, \bigwedge, Sym$ of strongly semistable vector bundles are also strongly semistable, so

$$E_{a_1...a_m, b_1...b_m} := E_1^{a_1} \otimes \cdots \otimes E_i^{a_i} \cdots \otimes E_m^{a_m} \otimes \bigwedge^{b_1} F_r^* E_1 \otimes \cdots \otimes \bigwedge^{b_i} F_r^* E_i \otimes \cdots \otimes \bigwedge^{b_m} F_r^* E_m$$

is a trival or strongly semistable direct sum part of $Sym^{a_1+...+a_m} E \otimes \bigwedge^{b_1+...+b_m} E$, for any non-negative integers $a_i, b_i, (1 \leq i \leq m)$. If $E_{a_1...a_m, b_1...b_m} \neq \emptyset$, then

$$\mu(E_{a_1...a_m, b_1...b_m}) = (a_1 + pb_1)\mu(E_1) + ... + (a_m + pb_m)\mu(E_m).$$

Using the natural homomorphism $F_r^*E_i \longrightarrow Sym^p(E_i)$, we can decompose the long exact sequence (4.2) into strongly semistable direct sum part according to $\mu = c_1\mu(E_1) + ... + c_m\mu(E_m), c_1 + ... + c_m = l$,

$$0 \longrightarrow (Sym^{l-l(p)p}(E) \otimes_k \bigwedge^{l(p)}(F_r^*E))_\mu \longrightarrow (Sym^{l-(l(p)-1)p}(E) \otimes_k \bigwedge^{l(p)-1}(F_r^*E))_\mu \longrightarrow$$

$$... \longrightarrow (Sym^{l-p}(E) \otimes_k \bigwedge(F_r^*E))_\mu \longrightarrow Sym^l(E)_\mu \longrightarrow T^l(E)_\mu \longrightarrow 0$$

So by Remark 5.3, $T^l(E)$ decompose into strongly semistable part $T^l(E)_\mu$ according to $\mu = c_1\mu(E_1) + ... + c_m\mu(E_m), c_1 + ... + c_m = l$.

Step 2: The computation of $I(T^l(E))$.

Let $r = rk(E), r_i = rk(E_i)$, we choose a local base $\{x_{r_1+..+r_{i-1}+1}, ..., x_{r_1+..+r_i}\}$ of E_i. For $l \le r(p-1), l = t(p-1) + s, 0 \le s < p-1$, write:

$$(d_1, ..., d_t, d_{t+1}, ..., d_r) = (p-1, .., p-1, s, ..., 0)$$

Then $\prod_{i=1}^r x_i^{d_i}$ $(\prod_{i=1}^r x_{r-i+1}^{d_i})$ belong to the maximum (minimum) slop stong semistable part of $T^l(E)$. Denote $\mu(x_i) = \mu(E_j)$, if $x_i \in E_j$, Then

$$\mu_{max}(T^lE) = \sum_{i=1}^r d_i\mu(x_i), \mu_{min}(T^lE) = \sum_{i=1}^r d_{r-i+1}\mu(x_i)$$

$$\mu_{max}(T^lE) - \mu_{min}(T^lE) = \sum_{i=1}^r (d_i - d_{r-i+1})\mu(x_i) = \sum_{i=1}^{[r/2]}(d_i - d_{r-i+1})(\mu(x_i) - \mu(x_{r-i+1}))$$

$$\le \sum_{i=1}^{[r/2]}(d_i - d_{r-i+1})(\mu_{max}(E) - \mu_{min}(E)) =$$

$$\begin{cases} lI(E) & l \le [r/2](p-1) \\ (r(p-1) - l)I(E) & r = 2k, r(p-1) > l > [r/2](p-1) \\ (r(p-1) - l + (d_{k+1} - (p-1)))I(E) & r = 2k+1, r(p-1) > l > [r/2](p-1) \\ 0 & l \ge r(p-1) \end{cases}$$

$\square$

Let E_1, E_2 be torsion free sheaves on X, The projective space

$$\mathbb{P} := \mathbb{P}(Ext_X^1(E_2, E_1)^\vee \oplus k)$$

parameterizes all extensions of E_1, E_2, including the trivial one $E_1 \oplus E_2$. And there is a tautological family

$$0 \longrightarrow q^* E_1 \otimes p^* \mathcal{O}_{\mathbb{P}}(1) \longrightarrow \mathcal{E} \longrightarrow q^* E_2 \longrightarrow 0$$

on $\mathbb{P} \times X$. Since any extension have the same Hilbert polynomial, so $\mathcal{E}$ is $\mathbb{P}$-flat.([2],p.186)

For any non-split extension

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

We can construct a torsion free sheaf $\mathcal{E}$ on $X \times \mathbb{P}^1$, such that $\mathcal{E}|_{X \times s} = E (s \neq 0)$ and $\mathcal{E}|_{X \times 0} = E_1 \oplus E_2$, we denote this $E \rightsquigarrow E_1 \oplus E_2$.

Proposition 5.5. *Let E be a torsion free sheaf on X,*

$$0 \subset E_1 \subset ... \subset E_m = E$$

be a filtration such that $E_i/E_{i-1} (1 \leq i \leq m)$ be torsion free sheaves, then

$$I(T^l(E)) \leq I(T^l(\oplus_{i=1}^m (E_i/E_{i-1}))).$$

Proof. We get a series of specialization as following

$$E_m \rightsquigarrow E_m/E_{m-1} \oplus E_{m-1} \rightsquigarrow ... \rightsquigarrow E_m/E_{m-1} \oplus E_{m-1}/E_{m-2} \oplus ... \oplus E_1$$

Thus construct a family $\mathcal{E}$ of torsion free sheaf, then $T^l \mathcal{E}$. There is a long exact sequence of $T^l \mathcal{E}$:

$$0 \longrightarrow Sym^{l-l(p)p}(\mathcal{E}) \otimes_k \bigwedge^{l(p)} (F_r^* \mathcal{E}) \longrightarrow Sym^{l-(l(p)-1)p}(\mathcal{E}) \otimes_k \bigwedge^{l(p)-1} (F_r^* \mathcal{E}) \longrightarrow ...$$

$$\longrightarrow Sym^{l-p}(\mathcal{E}) \otimes_k \bigwedge (F_r^* \mathcal{E}) \longrightarrow Sym^l(\mathcal{E}) \longrightarrow T^l(\mathcal{E}) \longrightarrow 0$$

$T^l(\mathcal{E})$ is also a flat family.

Since restrict functor is exact and commutes with tensor(symmetric,wedge,F_r^*) operations. By the upper semicontinuity of the Harder-Narasimhan polygons (Proposition 2.1), We have:

$$I(T^l(E)) = I((T^l \mathcal{E})_s) \leq I((T^l \mathcal{E})_{s_0}) = I(T^l(\oplus_{i=1}^{i=m} (E_i/E_{i-1}))).$$

$\square$

Theorem 5.6. *For any torsion free sheaf E of rank r, If $l \geq r(p-1)$, then $I(T^l E) = 0$. If $l < r(p-1)$, then*

$$I(T^l E) \leq Min\{l, [r/2](p-1)\}(I(E) + 2(r-1)L_X).$$

Proof. There is an integer k_0, such that if

$$0 \subset E_1 \subset ... \subset E_m = F^{k_0 *} E$$

is the Harder-Narasimhan filtration of $F^{k_0 *}E$, then $E_i/E_{i-1}(1 \leq i \leq m)$ are strongly semistable torsion free sheaves. Therefore,

$$\begin{array}{rcl} I(T^l E) & \leq & I(F^{k_0 *}(T^l E))/p^{k_0} \\ & = & I(T^l(F^{k_0 *}E))/p^{k_0} \\ & \leq & I(T^l(\oplus_{i=1}^m (E_i/E_{i-1})))/p^{k_0} \\ & \leq & Min\{l, [r/2](p-1)\} I(\oplus_{i=1}^m (E_i/E_{i-1}))/p^{k_0} \\ & = & Min\{l, [r/2](p-1)\}(L_{max}(E) - L_{min}(E)) \\ & \leq & Min\{l, [r/2](p-1)\}(I(E) + 2(r-1)L_X). \end{array}$$

The second and third inequalities follow from Proposition 5.5 and Proposition 5.4, respectively. This completes the proof. □

By Theorem 5.6 and Corolary 3.2, we have the following corollary.

Corollary 5.7. *Let W be a torsion free sheaf on X, then*

$$I(W, X) \leq I(W) + [n/2](p-1)I(\Omega_X) + (2(rk(E)-1) + 2(n-1)[n/2](p-1))L_X$$

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INSTABILITY OF TRUNCATED SYMMETRIC POWERS OF SHEAVES

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Abstract. Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic $p > 0$. Let $F_X : X \to X$ be the absolute Frobenius morphism, and $\mathscr{E}$ a torsion free sheaf on X. We give a upper bound of instability of truncated symmetric powers $\mathrm{T}^l(\mathscr{E})(0 \leq l \leq \mathrm{rk}(\mathscr{E})(p-1))$ in terms of $L_{\max}(\Omega^1_X)$, $\mathrm{I}(\Omega^1_X)$ and $\mathrm{I}(\mathscr{E})$ (Theorem 3.5). As an application, We obtain a upper bound of Frobenius direct image $F_{X*}(\mathscr{E})$ and some sufficient conditions of slope semi-stability of $F_{X*}(\mathscr{E})$. In addition, we study the slope (semi)-stability of sheaves of locally exact (closed) forms B^i_X (Z^i_X).

1. Introduction

Let k be an algebraically closed field k of characteristic $p > 0$, X a smooth projective variety of dimension n over k with a fixed ample divisor H. Let $\mathscr{E}$ be a torsion free sheaf on X, the slope of $\mathscr{E}$ is define as $\mu(\mathscr{E}) := \frac{c_1(\mathscr{E})\cdot H^{n-1}}{\mathrm{rk}(\mathscr{E})}$. Then $\mathscr{E}$ is called *slop* (*semi*)*-stable* if $\mu(\mathscr{F}) < (\leq)\mu(\mathscr{E})$ for any nonzero subsheaf $\mathscr{F} \subsetneq \mathscr{E}$ with $\mathrm{rk}(\mathscr{F}) < \mathrm{rk}(\mathscr{E})$. For any torsion free sheaf $\mathscr{E}$, there exists a unique filtration

$$\mathrm{HN}_\bullet(\mathscr{E}) : 0 = \mathrm{HN}_0(\mathscr{E}) \subset \mathrm{HN}_1(\mathscr{E}) \subset \cdots \subset \mathrm{HN}_{m-1}(\mathscr{E}) \subset \mathrm{HN}_m(\mathscr{E}) = \mathscr{E}$$

which is called *Harder-Narasimhan filtration* of $\mathscr{E}$, such that

(a). $gr^{\mathrm{HN}}_i(\mathscr{E}) := \mathrm{HN}_i(\mathscr{E})/\mathrm{HN}_{i-1}(\mathscr{E})(1 \leq i \leq m)$ are slope semistable.

(b). $\mu_{\max}(\mathscr{E}) := \mu(gr^{\mathrm{HN}}_1(\mathscr{E})) > \cdots > \mu(gr^{\mathrm{HN}}_{m-1}(\mathscr{E})) > \mu(gr^{\mathrm{HN}}_m(\mathscr{E})) =: \mu_{\min}(\mathscr{E})$.

The *instability* of $\mathscr{E}$ was defined as rational number

$$\mathrm{I}(\mathscr{E}) := \mu_{\max}(\mathscr{E}) - \mu_{\min}(\mathscr{E}),$$

which is a numerical index measures how far a torsion free sheaf from being slope semi-stable, and $\mathscr{E}$ is slope semi-stable if and only if $\mathrm{I}(\mathscr{E}) = 0$.

The absolute Frobenius morphism $F_X : X \to X$ is induced by homomorphism $\mathscr{O}_X \to \mathscr{O}_X$, $f \mapsto f^p$. If for any integer $m \geq 0$, m-th Frobenius pullback $F^{m*}_X(\mathscr{E})$ is slope (semi)-stable, then $\mathscr{E}$ is called a *slope strongly* (*semi*)*-stable sheaf*. In general, the slope (semi)-stability of torsion free sheaves does not preserved under Frobenius pull back F^*_X (cf. [1], [11]). However, V. Mehta, A. Ramanathan [9, Theorem 2.1] showed that if $\mu_{\max}(\Omega^1_X) \leq 0$ then all slope semi-stable sheaves on X are slope strongly semi-stable, and if $\mu_{\max}(\Omega^1_X) < 0$ then all slope stable sheaves on X are slope strongly stable. There are many classes of varieties satisfy $\mu_{\max}(\Omega^1_X) \leq 0$, such as homogeneous spaces, Abelian varieties, toric varieties and so on. The tensor product, exterior and symmetric products of slop strongly semi-stable sheaves are still slope strongly semi-stable in positive characteristic (cf. page 9 of [8]).

In general, A. Langer [7, Theorem 2.7] proved that for any torsion free sheaf $\mathscr{E}$ on X, there exists some integer $m_0 \geq 0$ such that all quotients of the Harder-Narasimhan filtration of $F_X^{m*}(\mathscr{E})$ are slope strongly semi-stable for any $m \geq m_0$. Moreover, A. Langer introduced the following definition

$$L_{\max}(\mathscr{E}) = \lim_{m\to\infty} \frac{\mu_{\max}(F_X^{m*}\mathscr{E})}{p^m},\ L_{\min}(\mathscr{E}) = \lim_{m\to\infty} \frac{\mu_{\min}(F_X^{m*}\mathscr{E})}{p^m},$$

and proved that [7, Corollary 6.2]

$$L_{\max}(\mathscr{E}) - L_{\min}(\mathscr{E}) \leq \frac{\mathrm{rk}(\mathscr{E})-1}{p} \cdot \max\{0,\ L_{\max}(\Omega_X^1)\} + \mathrm{I}(\mathscr{E}). \tag{1}$$

On the other hand, Frobenius push-forwards also does not preserve the slope semi-stability of torsion free sheaves. However, for a smooth projective curve of genus $g \geq 2$, V. Mehta and C. Pauly [10] showed that Frobenius direct images of slope semi-stable sheaves are also slope semi-stable. At the same time, X. Sun [14] proved that on a smooth projective curve X of genus $g \geq 1$, the Frobenius push-forwards preserves the slope semi-stability of locally free sheaves, and if genus $g \geq 2$ then Frobenius push-forwards preserves the slope stability of locally free sheaves via a different approach. In the case of higher dimension, X. Sun [14, Corollary 4.9] showed that: For any torsion free sheaf $\mathscr{E}$ on X, if $\mu(\Omega_X^1) \geq 0$, then

$$\mathrm{I}(F_{X*}(\mathscr{E})) \leq p^{n-1} \cdot \mathrm{rk}(\mathscr{E}) \cdot \max\{\mathrm{I}(\mathscr{E} \otimes_{\mathscr{O}_X} \mathrm{T}^l(\Omega_X^1)) \mid 0 \leq l \leq n(p-1)\}, \tag{2}$$

where $\mathrm{T}^l(\Omega_X^1)(0 \leq l \leq n(p-1))$ are truncated symmetric powers of Ω_X^1, which are the associated bundles of Ω_X^1 through some elementary representations of $\mathrm{GL}(n)$. Therefore, in order to bound the instability of Frobenius direct images, we should study the instabilities of $\mathrm{T}^l(\Omega_X^1)(0 \leq l \leq n(p-1))$.

More generally, we study the instability of truncated symmetric powers of any torsion free sheaf $\mathscr{E}$. One of the main results of this paper is to give a upper bound of $\mathrm{I}(\mathrm{T}^l(\mathscr{E}))$ in terms of $L_{\max}(\Omega_X^1)$, $\mathrm{I}(\Omega_X^1)$ and $\mathrm{I}(\mathscr{E})$ (Theorem 3.5).

As an application, we obtain a upper bound of Frobenius direct image $F_{X*}(\mathscr{E})$ in terms of $L_{\max}(\Omega_X^1)$, $\mathrm{I}(\Omega_X^1)$ and $\mathrm{I}(\mathscr{E})$ (Theorem 4.3) and some sufficient conditions of slope semi-stability of $F_{X*}(\mathscr{E})$ (when $\mu(\Omega_X^1) \geq 0$) (Proposition 4.4). In particular, when the cotangent sheaf Ω_X^1 is slope strongly semi-stable with $\mu(\Omega_X^1) \geq 0$, then the slope strongly semi-stability of $\mathscr{E}$ implies slope semi-stability of $F_{X*}(\mathscr{E})$.

In addition, we study the slope (semi)-stability of the sheaves of locally exact (closed) differential i-forms B_X^i (Z_X^i) on the higher dimensional smooth projective variety in positive characteristic. X. Sun [15] showed that when X is a smooth projective surface such that Ω_X^1 is slope semi-stable with $\mu(\Omega_X^1) > 0$, then Z_X^1 is slope semi-stable if $\mathrm{char}(k) = 3$ and Z_X^1 is slope stable if $\mathrm{char}(k) > 3$. However, we show that $Z_X^i(1 \leq i < \frac{n}{2})$ are never slope semi-stable if $n \geq 3$ and $\mu(\Omega_X^1) > 0$ (Proposition 5.2). Moreover, we show that if Ω_X^1 is slope semi-stable with $\mu(\Omega_X^1) = 0$, then B_X^i and Z_X^i are slope strongly semi-stable for any $1 \leq i \leq n$. At last, we prove that if $\mathrm{T}^l(\Omega_X^1)(0 \leq l \leq n(p-1))$ are slope semi-stable, then B_X^n is slope strongly semi-stable if $\mu(\Omega_X^1) = 0$ and B_X^n is slope stable if $\mu(\Omega_X^1) > 0$. This generalize the result of X. Sun [15, Lemma 3.2].

In this paper, unless otherwise explicitly declared, k is an algebraically closed field of characteristic $p > 0$, and X is a smooth projective variety of dimension n over k with a fixed ample divisor H.

2. Preliminaries

The canonical filtration was first introduced by Joshi-Ramanan-Xia-Yu in [4] in the curve case. X. Sun [14] generalized the canonical filtration to higher dimensional case and used this to study slope (semi)-stability of Frobenius direct images.

Definition 2.1. Let $\mathscr{E}$ be a coherent sheaf on X,

$$\nabla_{\mathrm{can}} : F_X^* F_{X*}(\mathscr{E}) \to F_X^* F_{X*}(\mathscr{E}) \otimes_{\mathscr{O}_X} \Omega_X^1$$

the canonical connection of $F_X^* F_{X*}(\mathscr{E})$ (cf. [5] Section 5). Set

$$\begin{aligned} V_1 &:= \mathrm{Ker}(F_X^* F_{X*}(\mathscr{E}) \twoheadrightarrow \mathscr{E})), \\ V_{l+1} &:= \mathrm{Ker}\{V_l \xrightarrow{\nabla} F_X^* F_{X*}(\mathscr{E}) \otimes_{\mathscr{O}_X} \Omega_X^1 \to (F_X^* F_{X*}(\mathscr{E})/V_l) \otimes_{\mathscr{O}_X} \Omega_X^1\}. \end{aligned}$$

The filtration $\mathbb{F}^{\mathrm{can}}_{\mathscr{E}\,\bullet} : F_X^* F_{X*}(\mathscr{E}) = V_0 \supset V_1 \supset V_2 \supset \cdots$, is called the *canonical filtration* of $F_X^* F_{X*}(\mathscr{E})$.

Theorem 2.2. [14, Theorem 3.7] *Let $\mathscr{E}$ be a locally free sheaf on X. Then the canonical filtration of $F_X^* F_{X*}(\mathscr{E})$ is*

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = F_X^* F_{X*}(\mathscr{E})$$

such that $\nabla^l : V_l/V_{l+1} \cong \mathscr{E} \otimes_{\mathscr{O}_X} \mathrm{T}^l(\Omega_X^1)$, $0 \le l \le n(p-1)$, where $\mathrm{T}^l(\Omega_X^1)$ are the truncated symmetric powers of Ω_X^1.

We recall the construction and properties of truncated symmetric powers of vector spaces. Let V is a n-dimensional vector space over k with standard representation of $\mathrm{GL}_n(k)$, S_l the symmetric group of l elements with a natural action on $V^{\otimes l}$ by $(v_1 \otimes \cdots \otimes v_l) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(l)}$ for $v_i \in V$ and $\sigma \in S_l$. Let $e_1, \cdots, e_n$ be a basis of V. For any non-negative partition $(k_1, \cdots, k_n)$ of l (i.e. $l = \sum_{i=1}^{n} k_i$, $k_i \ge 0$, $1 \le i \le n$), define $v(k_1, \cdots, k_n) := \sum_{\sigma \in S_l} (e_1^{\otimes k_1} \otimes \cdots \otimes e_n^{\otimes k_n}) \cdot \sigma$. Let $\mathrm{T}^l(V) \subset V^{\otimes l}$ be the linear subspace generated by all vectors

$$\{v(k_1, \cdots, k_n) \mid l = \sum_{i=1}^{n} k_i,\ k_i \ge 0,\ 1 \le i \le n\}.$$

Then $\mathrm{T}^l(V)$ is a $\mathrm{GL}_n(k)$ sub-representation of $V^{\otimes l}$. Let $F_k^* V$ be the Frobenius twist of the standard representation V of $\mathrm{GL}_n(k)$ through the homomorphism $\mathrm{GL}_n(k) \to \mathrm{GL}_n(k) : ((a_{ij})_{n\times n} \mapsto (a_{ij}^p)_{n \times n})$. Define k-linear maps

$$\phi_q : \mathrm{Sym}^{l-q\cdot p}(V) \otimes_k \bigwedge^{q}(F_k^* V) \to \mathrm{Sym}^{l-(q-1)\cdot p}(V) \otimes_k \bigwedge^{q-1}(F_k^* V) \tag{3}$$

$$f \otimes e_{k_1} \wedge \cdots \wedge e_{k_q} \mapsto \sum_{i=1}^{q} (-1)^{i-1} e_{k_i}^p f \otimes e_{k_1} \wedge \cdots \wedge \widehat{e}_{k_i} \wedge \cdots \wedge e_{k_q}$$

Lemma 2.3. [14, Proposition 3.5] *There exists an exact sequence of $\mathrm{GL}_n(k)$-representations*

$$\begin{aligned} 0 \to \mathrm{Sym}^{l-l(p)\cdot p}(V) \otimes_k \bigwedge^{l(p)} F_k^*(V) \xrightarrow{\phi_{l(p)}} \mathrm{Sym}^{l-(l(p)-1)\cdot p}(V) \otimes_k \bigwedge^{l(p)-1} F_k^*(V) \to \\ \cdots \to \mathrm{Sym}^{l-q\cdot p}(V) \otimes_k \bigwedge^{q} F_k^*(V) \xrightarrow{\phi_q} \mathrm{Sym}^{l-(q-1)\cdot p}(V) \otimes_k \bigwedge^{q-1} F_k^*(V) \to \cdots \\ \to \mathrm{Sym}^{l-p}(V) \otimes_k F_k^*(V) \xrightarrow{\phi_1} \mathrm{Sym}^{l}(V) \xrightarrow{\phi_0} \mathrm{T}^l(V) \to 0. \end{aligned}$$

Let $\mathscr{E}$ be a locally free sheaf of rank n on X, $\mathrm{T}^l(\mathscr{E}) \subset \mathscr{E}^{\otimes l}$ is defined to be the sheaf of sections of the associated vector bundle of the frame bundle of $\mathscr{E}$ (principal $\mathrm{GL}_n(k)$-bundle) through the representation $\mathrm{T}^l(V)$.

3. Instability of Truncated Symmetric Powers

Let we recall the definition and some properties of Harder-Narasimhan polygons of sheaves, which was first introduced by S. S. Shatz in [12].

For any coherent sheaf $\mathscr{F}$ on X, we may associate to it the point $\mathrm{p}(\mathrm{rk}(\mathscr{F}), \deg(\mathscr{F}))$ in the plane with coordinates rank and degree. Let $\mathscr{E}$ be a torsion free sheaf,

$$\mathrm{HN}_{\bullet}(\mathscr{E}) : 0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{m-1} \subset \mathscr{E}_m = \mathscr{E}$$

the Harder-Narasimhan filtration of $\mathscr{E}$. Consider the points

$$\mathrm{p}(\mathrm{rk}(\mathscr{E}_i), \deg(\mathscr{E}_i))(1 \leq i \leq m)$$

in the coordinate plane rank-deg, and connect point $\mathrm{p}(\mathrm{rk}(\mathscr{E}_i), \deg(\mathscr{E}_i))$ to point $\mathrm{p}(\mathrm{rk}(\mathscr{E}_{i+1}), \deg(\mathscr{E}_{i+1}))$ successively by line segments. Then we get a polygon in the plane which we call the *Harder-Narasimhan Polygon* of $\mathscr{E}$, denote by $\mathrm{HNP}(\mathscr{E})$.

Let $\mathrm{p}(r, d)$ be a point in the coordinate plane of rank-deg. If $r \leq \mathrm{rk}(\mathscr{E})$, and $d \geq (\leq) d'$ for some point $\mathrm{p}(r, d') \in \mathrm{HNP}(\mathscr{E})$, then we say $\mathrm{p}(r, d)$ *lies on* (*below*) the $\mathrm{HNP}(\mathscr{E})$. In this sense, there is a natural partial order structure, denote by "$\succcurlyeq$", on the set $\{\mathrm{HNP}(\mathscr{E}) \mid \mathscr{E} \in \mathfrak{Coh}(X)\ \}$.

Lemma 3.1 (S. S. Shatz [12] Theorem 3)**.** *Let k be an algebraically closed field, S a scheme of finite type over k, $\mathcal{E}$ a flat family of torsion free sheaves on $S \times_k X$ parameterized by S. Then for any convex polygon $\mathscr{P}$, subset $S_{\mathscr{P}} = \{s \in S | \mathrm{HNP}(\mathcal{E}|_s) \succcurlyeq \mathscr{P}\}$ is a closed subset of S.*

Let $\mathscr{E}$ a torsion free sheaf on X. Then there exists an open subset $U \subseteq X$, such that $\mathrm{codim}_X(X - U) \geq 2$ and $\mathscr{E}|_U$ is locally free. Let

$$\widehat{\mathrm{T}^l(\mathscr{E})}_U := i_{U*}(\mathrm{T}^l(\mathscr{E}|_U))$$

where $i_U : U \to X$ is the natural open immersion. Then $\widehat{\mathrm{T}^l(\mathscr{E})}_U$ is a reflective torsion free sheaf such that $\mu(\widehat{\mathrm{T}^l(\mathscr{E})}_U)$ and $\mathrm{I}(\widehat{\mathrm{T}^l(\mathscr{E})}_U)$ are independent of the choice of U. Without loss of generality, for any torsion free sheaf $\mathscr{E}$, we denote

$$\mu(\mathrm{T}^l(\mathscr{E})) := \mu(\widehat{\mathrm{T}^l(\mathscr{E})}_U),\ \mathrm{I}(\mathrm{T}^l(\mathscr{E})) := \mathrm{I}(\widehat{\mathrm{T}^l(\mathscr{E})}_U),$$

for any open subset $U \subseteq X$ such that $\mathrm{codim}_X(X - U) \geq 2$ and $\mathscr{E}|_U$ is locally free.

Lemma 3.2. *Let R be a principal ideal domain, $S = \mathrm{Spec}(R)$, and X an integral scheme over S, U an open subscheme of X. Let $\mathscr{E}$ be a torsion free sheaf on U which is flat over S such that $i_{U*}(\mathscr{E})$ is a coherent sheaf on X, where $i_U : U \to X$ is the natural open immersion. Then $i_{U*}(\mathscr{E})$ is flat over S.*

Proof. Since R is a principal ideal domain, to prove $i_{U*}(\mathscr{E})$ is flat over S, it is enough to show that $M := (i_{U*}(\mathscr{E}))(V)$ is a torsion free R-module for any affine open subset $V \subseteq X$. Let $0 \neq r \in R$, $0 \neq m \in M$. As X is an integral scheme and $\mathscr{E}$ is torsion free over U, then $m|_W \neq 0$ for any affine open sub-set $W \subseteq U \cap V$. On the other hand, $r \cdot (m|_W) \neq 0$ for $\mathscr{E}$ is flat over S, therefore $rm \neq 0$. Thus M is a torsion free R-module. □

Proposition 3.3. *Let $\mathscr{E}$ be a torsion free sheaf on X, $0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{m-1} \subset \mathscr{E}_m = \mathscr{E}$ a filtration of $\mathscr{E}$ such that $\mathscr{E}_i/\mathscr{E}_{i-1} (1 \leq i \leq m)$ are torsion free sheaves. Then for any integer $0 \leq l \leq \mathrm{rk}(\mathscr{E})(p-1)$, we have*

$$\mathrm{I}(\mathrm{T}^l(\mathscr{E})) \leq \mathrm{I}(\mathrm{T}^l(\bigoplus_{i=1}^m \mathscr{E}_i/\mathscr{E}_{i-1})).$$

Proof. We use induction on m, so we only have to prove the case $m=2$. In fact, for any $1 \le i \le m$, consider the exact sequence

$$0 \longrightarrow \mathscr{E}_{i-1} \bigoplus (\bigoplus_{j=i+1}^{m} \mathscr{E}_j/\mathscr{E}_{j-1}) \longrightarrow \mathscr{E}_i \bigoplus (\bigoplus_{j=i+1}^{m} \mathscr{E}_j/\mathscr{E}_{j-1}) \longrightarrow \mathscr{E}_i/\mathscr{E}_{i-1} \longrightarrow 0,$$

we have $\mathrm{I}(\mathrm{T}^l(\mathscr{E})) \le \mathrm{I}(\mathrm{T}^l(\mathscr{E}_i \bigoplus (\bigoplus_{j=i+1}^{m} \mathscr{E}_j/\mathscr{E}_{j-1}))) \le \mathrm{I}(\mathrm{T}^l(\bigoplus_{i=1}^{m} \mathscr{E}_i/\mathscr{E}_{i-1}))$.

In general, for the exact sequence of torsion free sheaves

$$0 \longrightarrow \mathscr{E}' \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}'' \longrightarrow 0,$$

we can construct a coherent sheaf $\mathcal{E}$ on $\mathbb{A}^1_k \times X$ which is flat over $\mathbb{A}^1_k$, such that $\mathcal{E}|_{t\times X} \cong \mathscr{E}$ for any $t \in \mathbb{A}^1_k - \{0\}$, and $\mathcal{E}|_{\{0\}\times X} \cong \mathscr{E}' \bigoplus \mathscr{E}''$. Choose an open subset $U \subseteq X$ with $\mathrm{codim}_X(X-U) \ge 2$ such that $\mathscr{E}|_U$, $\mathscr{E}'|_U$ and $\mathscr{E}''|_U$ are locally free. Then for any $t \in \mathbb{A}^1_k$, $\mathcal{E}|_{\{t\}\times U}$ is locally free on U. Hence, by [13, Theorem 1.27] we have $\mathcal{E}|_{\mathbb{A}^1_k \times U}$ is locally free on $\mathbb{A}^1_k \times U$.

By Lemma 2.3 we can construct an exact sequence of sheaves on $Y := \mathbb{A}^1_k \times U$

$$\begin{aligned} 0 \to \mathrm{Sym}^{l-l(p)\cdot p}(\mathcal{E}|_Y) \otimes_{\mathscr{O}_Y} \bigwedge^{l(p)} F_Y^*(\mathcal{E}|_Y) \xrightarrow{\phi_{l(p)}} \mathrm{Sym}^{l-(l(p)-1)\cdot p}(\mathcal{E}|_Y) \otimes_{\mathscr{O}_Y} \bigwedge^{l(p)-1} F_Y^*(\mathcal{E}|_Y) \to \\ \cdots \to \mathrm{Sym}^{l-q\cdot p}(\mathcal{E}|_Y) \otimes_{\mathscr{O}_Y} \bigwedge^{q} F_Y^*(\mathcal{E}|_Y) \xrightarrow{\phi_q} \mathrm{Sym}^{l-(q-1)\cdot p}(\mathcal{E}|_Y) \otimes_{\mathscr{O}_Y} \bigwedge^{q-1} F_Y^*(\mathcal{E}|_Y) \to \cdots \\ \to \mathrm{Sym}^{l-p}(\mathcal{E}|_Y) \otimes_{\mathscr{O}_Y} F_Y^*(\mathcal{E}|_Y) \xrightarrow{\phi_1} \mathrm{Sym}^{l}(\mathcal{E}|_Y) \xrightarrow{\phi_0} \mathrm{T}^l(\mathcal{E}|_Y) \to 0. \end{aligned}$$

Let

$$\widehat{\mathrm{T}^l(\mathcal{E})} := {i_{\mathbb{A}^1_k \times U}}_*(\mathrm{T}^l(\mathcal{E}|_{\mathbb{A}^1_k \times U}))$$

where $i_{\mathbb{A}^1_k \times U} : \mathbb{A}^1_k \times U \to \mathbb{A}^1_k \times X$ is the natural open immersion. Since $\mathbb{A}^1_k \times X$ is a smooth variety and $\mathrm{codim}_{\mathbb{A}^1_k \times X}(\mathbb{A}^1_k \times X - \mathbb{A}^1_k \times U) \ge 2$, $\mathrm{T}^l(\mathcal{E}|_{\mathbb{A}^1_k \times U})$ is locally free, by [2, Proposition 5.10] and Lemma 3.2, we have $\widehat{\mathrm{T}^l(\mathcal{E})}$ is a coherent sheaf on $\mathbb{A}^1_k \times X$ which is flat over $\mathbb{A}^1_k$ such that $\widehat{\mathrm{T}^l(\mathcal{E})}_t|_U \cong \mathrm{T}^l(\mathscr{E}|_U)$ for any $t \in \mathbb{A}^1_k - \{0\}$ and $\widehat{\mathrm{T}^l(\mathcal{E})}_0|_U \cong \mathrm{T}^l((\mathscr{E}' \bigoplus \mathscr{E}'')|_U)$. Then by Lemma 3.1, we have

$$\mathrm{I}(\mathrm{T}^l(\mathscr{E})) = \mathrm{I}(\widehat{\mathrm{T}^l(\mathcal{E})}_t) \le \mathrm{I}(\widehat{\mathrm{T}^l(\mathcal{E})}_0) = \mathrm{I}(\mathrm{T}^l(\mathscr{E}' \oplus \mathscr{E}'')).$$

for any $t \in \mathbb{A}^1_k - \{0\}$. This completes the proof. □

The following theorem shows that when $\mathscr{E}$ is a direct sum of slope strongly semi-stable sheaves, we can bound $\mathrm{I}(\mathrm{T}^l(\mathscr{E}))$ in terms of $\mathrm{rk}(\mathscr{E})$ and $\mathrm{I}(\mathscr{E})$.

Theorem 3.4. *Let $\mathscr{E} = \mathscr{E}_1 \oplus \cdots \oplus \mathscr{E}_m$ be a torsion free sheaf such that $\mathscr{E}_i$ $(1 \le i \le m)$ are slope strongly semi-stable. Then for any integer $0 \le l \le \mathrm{rk}(\mathscr{E})(p-1)$, we have*

$$\mathrm{I}(\mathrm{T}^l(\mathscr{E})) \le \min\{\ l,\ [\frac{\mathrm{rk}(\mathscr{E})}{2}](p-1)\ \} \cdot \mathrm{I}(\mathscr{E}).$$

Proof. Without loss of generality, we can assume $\mathscr{E}_i (1 \le i \le m)$ are locally free such that $\mu(\mathscr{E}_1) > \mu(\mathscr{E}_2) > \cdots > \mu(\mathscr{E}_m)$. Let $r = \mathrm{rk}(\mathscr{E})$, $r_i = \mathrm{rk}(\mathscr{E}_i)(1 \le i \le m)$, $l(p)$ be the unique integer such that $0 \le l - l(p)\cdot p < p$.

Since tensor products, exterior and symmetric products of slope strongly semi-stable sheaves are still slope strongly semi-stable, the direct summand

$$\mathscr{E}^{a_1,\cdots,a_m}_{b_1,\cdots,b_m} := \mathrm{Sym}^{a_1}\mathscr{E}_1 \otimes \cdots \otimes \mathrm{Sym}^{a_m}\mathscr{E}_m \otimes \bigwedge^{b_1} F_X^*\mathscr{E}_1 \otimes \cdots \otimes \bigwedge^{b_m} F_X^*\mathscr{E}_m$$

of $\mathrm{Sym}^a\mathscr{E} \otimes_{\mathscr{O}_X} \bigwedge^{b} F_X^*\mathscr{E}$ is slope strongly semi-stable for any non-negative partition $(a_1,\cdots,a_m)$, $(b_1,\cdots,b_m)$ of non-negative integers a and b. By direct computation, if $\mathscr{E}^{a_1,\cdots,a_m}_{b_1,\cdots,b_m} \ne \emptyset$, then $\mu(\mathscr{E}^{a_1,\cdots,a_m}_{b_1,\cdots,b_m}) = (a_1 + pb_1)\mu(\mathscr{E}_1) + \cdots + (a_m + pb_m)\mu(\mathscr{E}_m)$.

By Lemma 2.3, we obtain the exact sequence of locally free sheaves

$$0 \to \mathrm{Sym}^{l-l(p)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{l(p)} F_X^*(\mathscr{E}) \xrightarrow{\phi_{l(p)}} \mathrm{Sym}^{l-(l(p)-1)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{l(p)-1} F_X^*(\mathscr{E}) \to$$
$$\cdots \to \mathrm{Sym}^{l-q\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{q} F_X^*(\mathscr{E}) \xrightarrow{\phi_q} \mathrm{Sym}^{l-(q-1)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{q-1} F_X^*(\mathscr{E}) \to \cdots$$
$$\to \mathrm{Sym}^{l-p}(\mathscr{E}) \otimes_{\mathscr{O}_X} F_X^*(\mathscr{E}) \xrightarrow{\phi_1} \mathrm{Sym}^{l}(\mathscr{E}) \xrightarrow{\phi_0} \mathrm{T}^l(\mathscr{E}) \to 0. \tag{4}$$

By the definition of ϕ_q in section 2 (See (3)), we have

$$\phi_q(\mathscr{E}^{a_1,\cdots,a_m}_{b_1,\cdots,b_m}) \subseteq \bigoplus_{i=1}^{m} \mathscr{E}^{a_1,\cdots,a_i+p,\cdots a_m}_{b_1,\cdots,b_i-1,\cdots,b_m}.$$

where $l = \sum\limits_{i=1}^{m}(a_i+pb_i)$. As $\mu(\mathscr{E}^{a_1,\cdots,a_i+p,\cdots,a_m}_{b_1,\cdots,b_i-1,\cdots,b_m}) = \bigoplus\limits_{i=1}^{m}(a_i+pb_i)\mu(\mathscr{E}_i)$ for any $1 \leq i \leq m$. Thus $\mu(\mathscr{E}^{a_1,\cdots,a_m}_{b_1,\cdots,b_m}) = \mu(\bigoplus\limits_{i=1}^{m} \mathscr{E}^{a_1,\cdots,a_i+p,\cdots a_m}_{b_1,\cdots,b_i-1,\cdots,b_m})$. Let $(\mathrm{Sym}^{l-q\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge\limits^{q} F_X^*(\mathscr{E}))_\mu$ be the direct summand of $\mathrm{Sym}^{l-q\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^q F_X^*(\mathscr{E})$ with slope μ, where $\mu = \sum\limits_{i=1}^{m} c_i\mu(\mathscr{E}_i)$, $l = \sum\limits_{i=1}^{m} c_i$, $c_i \geq 0 (0 \leq i \leq m)$ and $0 \leq q \leq l(p)$. Then the exact sequence (4) can be decomposed into direct summands

$$0 \to (\mathrm{Sym}^{l-l(p)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{l(p)} F_X^*(\mathscr{E}))_\mu \xrightarrow{\phi_{l(p)}} (\mathrm{Sym}^{l-(l(p)-1)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{l(p)-1} F_X^*(\mathscr{E}))_\mu \to$$
$$\cdots \to (\mathrm{Sym}^{l-q\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{q} F_X^*(\mathscr{E}))_\mu \xrightarrow{\phi_q} (\mathrm{Sym}^{l-(q-1)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{q-1} F_X^*(\mathscr{E}))_\mu \to \cdots$$
$$\to (\mathrm{Sym}^{l-p}(\mathscr{E}) \otimes_{\mathscr{O}_X} F_X^*(\mathscr{E}))_\mu \xrightarrow{\phi_1} (\mathrm{Sym}^{l}(\mathscr{E}))_\mu \xrightarrow{\phi_0} (\mathrm{T}^l(\mathscr{E}))_\mu \to 0,$$

and

$$\mathrm{T}^l(\mathscr{E}) = \bigoplus_{\substack{\mu=\sum_{i=1}^{m} c_i\mu(\mathscr{E}_i),\\ \sum_{i=1}^{m} c_i=l,\, c_i\geq 0 (1\leq i\leq m)}} (\mathrm{T}^l(\mathscr{E}))_\mu.$$

Consider the direct sum $\mathrm{Sym}^l(\mathscr{E}) = \bigoplus\limits_{\substack{\sum_{i=1}^{m} k_i=l,\\ k_i\geq 0, 1\leq i\leq m}} \bigotimes\limits_{i=1}^{m} \mathrm{Sym}^{k_i}(\mathscr{E}_i)$, we have

$$\phi_0(\bigotimes_{i=1}^{m} \mathrm{Sym}^{k_i}(\mathscr{E}_i)) \neq 0 \text{ if and only if } 0 \leq k_i < r_i(p-1) (1 \leq i \leq m).$$

Let $(d_1, \cdots, d_{l(p)}, \cdots, d_r) := (p-1, \cdots, p-1, l-l(p)\cdot p, \cdots, 0)$. Then

$$\mu_{\max}(\mathrm{T}^l(\mathscr{E})) = \sum_{i=1}^{r} d_i\mu(\mathscr{E}_i),\ \mu_{\min}(\mathrm{T}^l(\mathscr{E})) = \sum_{i=1}^{r} d_{r-i+1}\mu(\mathscr{E}_i).$$

$$\begin{aligned} \mathrm{I}(\mathrm{T}^l(\mathscr{E})) &= \sum_{i=1}^{r}(d_i - d_{r-i+1})\mu(\mathscr{E}_i) = \sum_{i=1}^{[\frac{r}{2}]}(d_i - d_{r-i+1})(\mu(\mathscr{E}_i) - \mu(\mathscr{E}_{r-i+1})) \\ &\leq \sum_{i=1}^{[\frac{r}{2}]}(d_i - d_{r-i+1})(\mu_{\max}(\mathscr{E}) - \mu_{\min}(\mathscr{E})). \end{aligned}$$

$$= \begin{cases} l \cdot \mathrm{I}(\mathscr{E}) & \text{if } 0 \le l \le [\frac{r}{2}](p-1) \\ (r(p-1)-l) \cdot \mathrm{I}(\mathscr{E}) & \text{if } r|2, \frac{r}{2}(p-1) < l < r(p-1) \\ (p-1) \cdot [\frac{r}{2}] \cdot \mathrm{I}(\mathscr{E}) & \text{if } r \nmid 2, [\frac{r}{2}](p-1) < l \le ([\frac{r}{2}]+1)(p-1) \\ (r(p-1)-l) \cdot \mathrm{I}(\mathscr{E}) & \text{if } r \nmid 2, ([\frac{r}{2}]+1)(p-1) < l < r(p-1) \end{cases}$$

Hence, $\mathrm{I}(\mathrm{T}^l(\mathscr{E})) \le \min\{\ l,\ [\frac{\mathrm{rk}(\mathscr{E})}{2}](p-1)\ \} \cdot \mathrm{I}(\mathscr{E})$, for any $0 \le l \le \mathrm{rk}(\mathscr{E})(p-1)$. □

Theorem 3.5. *Let $\mathscr{E}$ be a torsion free sheaf on X. Then for any integer $0 \le l \le \mathrm{rk}(\mathscr{E})(p-1)$, we have*

$$\mathrm{I}(\mathrm{T}^l(\mathscr{E})) \le \min\{\ l,\ [\frac{\mathrm{rk}(\mathscr{E})}{2}](p-1)\ \} \cdot (\frac{\mathrm{rk}(\mathscr{E})-1}{p} \cdot \max\{0,\ L_{\max}(\Omega^1_X)\} + \mathrm{I}(\mathscr{E})).$$

Proof. By [7, Theorem 2.7], there exists an integer $m_0 \ge 0$ such that the Harder-Narasimhan filtration $0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_{m-1} \subset \mathscr{E}_m = F_X^{m_0\,*}(\mathscr{E})$ of $F_X^{m_0\,*}(\mathscr{E})$ satisfy $gr_i^{\mathrm{HN}}(F_X^{m_0\,*}(\mathscr{E})) = \mathscr{E}_i/\mathscr{E}_{i-1}(1 \le i \le m)$ are slope strongly semi-stable. Thus,

$$\begin{aligned} \mathrm{I}(\mathrm{T}^l(\mathscr{E})) &\le \frac{\mathrm{I}(F_X^{m_0*}(\mathrm{T}^l(\mathscr{E}))}{p^{m_0}} \\ &= \frac{\mathrm{I}(\mathrm{T}^l(F_X^{m_0*}(\mathscr{E})))}{p^{m_0}} \\ &\le \frac{\mathrm{I}(\mathrm{T}^l(\bigoplus_{i=1}^{m} \mathscr{E}_i/\mathscr{E}_{i-1}))}{p^{m_0}} \text{ (cf. Theorem 3.3)} \\ &\le \min\{\ l,\ [\frac{\mathrm{rk}(\mathscr{E})}{2}](p-1)\ \} \cdot \frac{\mathrm{I}(\bigoplus_{i=1}^{m} \mathscr{E}_i/\mathscr{E}_{i-1})}{p^{m_0}} \text{ (cf. Theorem 3.4)} \\ &= \min\{\ l,\ [\frac{\mathrm{rk}(\mathscr{E})}{2}](p-1)\ \} \cdot (L_{\max}(\mathscr{E}) - L_{\min}(\mathscr{E})) \\ &\le \min\{\ l,\ [\frac{\mathrm{rk}(\mathscr{E})}{2}](p-1)\ \} \cdot (\frac{\mathrm{rk}(\mathscr{E})-1}{p} \cdot \max\{0,\ L_{\max}(\Omega^1_X)\} + \mathrm{I}(\mathscr{E})). \end{aligned}$$

where the first inequality is followed by the property of Frobenius morphism, and the last inequality is just the inequality (1). □

We give some sufficient conditions for slope semi-stability of truncated symmetric powers of torsion free sheaves as following.

Proposition 3.6. *Let $\mathscr{E}$ be a slope strongly semi-stable torsion free sheaf on X. Then $\mathrm{T}^l(\mathscr{E})$ is slope strongly semi-stable for any integer $0 \le l \le \mathrm{rk}(\mathscr{E})(p-1)$.*

Proof. Without loss of generality, we can assume $\mathscr{E}$ is locally free. For any integer $0 \le l \le \mathrm{rk}(\mathscr{E})(p-1)$, consider the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Sym}^{l-l(p)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{l(p)} F_X^*(\mathscr{E}) \longrightarrow \mathrm{Sym}^{l-(l(p)-1)\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{l(p)-1} F_X^*(\mathscr{E}) \longrightarrow \cdots \\ \longrightarrow \mathrm{Sym}^{l-p}(\mathscr{E}) \otimes_{\mathscr{O}_X} F_X^*(\mathscr{E}) \longrightarrow \mathrm{Sym}^l(\mathscr{E}) \longrightarrow \mathrm{T}^l(\mathscr{E}) \longrightarrow 0, \end{aligned}$$

we have $\mu(\mathrm{Sym}^{l-i\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{i} F_X^*(\mathscr{E})) = \mu(\mathrm{T}^l(\mathscr{E}))$ for any integer $0 \le i \le l(p)$ by direct computation. Since $\mathrm{Sym}^{l-i\cdot p}(\mathscr{E}) \otimes_{\mathscr{O}_X} \bigwedge^{i} F_X^*(\mathscr{E})(0 \le i \le l(p))$ are slope strongly semi-stable. Thus $\mathrm{T}^l(\mathscr{E})$ is also slope strongly semi-stable by the trivial remark: For any short exact sequence of torsion free sheaves with same slope, the middle term is slope (strongly) semi-stable if and only if the other two terms are slope (strongly) semi-stable. □

4. Instability of Frobenius Direct Images

In this section, we study the instability of Frobenius direct images of torsion free sheaves in two cases. (I). Ω^1_X is slope semi-stable with $\mu(\Omega^1_X)\leq 0$; (II). $\mu(\Omega^1_X)\geq 0$.

4.1. Case I: Ω^1_X is slope semi-stable with $\mu(\Omega^1_X)\leq 0$.

Theorem 4.1. *If Ω^1_X is slope semi-stable with $\mu(\Omega^1_X)\leq 0$. Then for any slope semi-stable sheaf $\mathscr{E}$ on X, we have*

$$\mathrm{I}(F_{X*}(\mathscr{E}))\leq -\frac{n(p-1)p^{n-1}\cdot\mathrm{rk}(\mathscr{E})\cdot\mu(\Omega^1_X)}{2}.$$

Proof. Since Ω^1_X is slope semi-stable with $\mu(\Omega^1_X)\leq 0$, hence $\mathscr{E}\otimes_{\mathscr{O}_X}\mathrm{T}^l(\Omega^1_X)$ is slope semi-stable for any $0\leq l\leq n(p-1)$. Let $0\subset V_{n(p-1)}\subset\cdots\subset V_1\subset V_0=F^*_XF_{X*}(\mathscr{E})$ be the canonical filtration of $F^*_XF_{X*}(\mathscr{E})$. For any sub-sheaf $\mathscr{F}$ of $F_{X*}(\mathscr{E})$, let $m=\max\{\ l\mid V_m\cap F^*_{X/k}(\mathscr{E})\neq 0\ \}$ and

$$\mathscr{E}_l:=\frac{V_l\cap F^*_{X/k}(\mathscr{E})}{V_{l+1}\cap F^*_{X/k}(\mathscr{E})}\subset\frac{V_l}{V_{l+1}},\ r_l:=\mathrm{rk}(\mathscr{E}_l),\ 0\leq l\leq m.$$

Then by [14, Lemma 4.4], we have

$$\begin{aligned}\mu(\mathscr{F})-\mu(F_{X*}(\mathscr{E})) &\leq -\frac{\mu(\Omega^1_X)}{p\cdot\mathrm{rk}(\mathscr{F})}\sum_{l=0}^{m}(\frac{n(p-1)}{2}-l)\cdot r_l\\ &\leq -\frac{n(p-1)\cdot\mu(\Omega^1_X)}{2\cdot p}.\end{aligned}$$

Thus

$$\mathrm{I}(F_{X*}(\mathscr{E}))\leq -\frac{n(p-1)\cdot\mu(\Omega^1_X)}{2\cdot p}\cdot\mathrm{rk}(F_{X*}(\mathscr{E}))=-\frac{n(p-1)p^{n-1}\cdot\mathrm{rk}(\mathscr{E})\cdot\mu(\Omega^1_X)}{2}.$$

This is follows from the fact that for any torsion free sheaf $\mathscr{F}$, if there is a constant λ such that $\mu(\mathscr{G})-\mu(\mathscr{F})\leq\lambda$ for any subsheaf $\mathscr{G}\subset\mathscr{F}$. Then $\mathrm{I}(\mathscr{F})\leq\mathrm{rk}(\mathscr{F})\cdot\lambda$. □

4.2. Case II: $\mu(\Omega^1_X)\geq 0$.

Proposition 4.2. *Let $\mathscr{E}_i(1\leq i\leq m)$ be torsion free sheaves. Then*

$$\mathrm{I}(\bigotimes_{i=1}^m\mathscr{E}_i)\leq\frac{\sum\limits_{i=1}^m\mathrm{rk}(\mathscr{E}_i)-m}{p}\cdot\max\{0,\ L_{\max}(\Omega^1_X)\}+\sum_{i=1}^m\mathrm{I}(\mathscr{E}_i).$$

Proof. Since $L_{\max}(\bigotimes\limits_{i=1}^m\mathscr{E}_i)=\sum\limits_{i=1}^m L_{\max}(\mathscr{E}_i)$, we have

$$L_{\min}(\bigotimes_{i=1}^m\mathscr{E}_i)=-L_{\max}((\bigotimes_{i=1}^m\mathscr{E}_i)^\vee)=-L_{\max}(\bigotimes_{i=1}^m\mathscr{E}_i^\vee)=\sum_{i=1}^m L_{\min}(\mathscr{E}_i).$$

Hence,

$$\begin{aligned}\mathrm{I}(\bigotimes_{i=1}^m\mathscr{E}_i) &\leq L_{\max}(\bigotimes_{i=1}^m\mathscr{E}_i)-L_{\min}(\bigotimes_{i=1}^m\mathscr{E}_i)=\sum_{i=1}^m(L_{\max}(\mathscr{E}_i)-L_{\min}(\mathscr{E}_i))\\ &\leq\frac{\sum\limits_{i=1}^m\mathrm{rk}(\mathscr{E}_i)-m}{p}\cdot\max\{0,\ L_{\max}(\Omega^1_X)\}+\sum_{i=1}^m\mathrm{I}(\mathscr{E}_i)\end{aligned}$$

□

Theorem 4.3. *If $\mu(\Omega_X^1) \geq 0$. Then for any torsion free sheaf $\mathscr{E}$ on X, we have*

$$\begin{aligned}
\mathrm{I}(F_{X*}(\mathscr{E})) \quad \leq \quad & \left\{ \frac{\max\limits_{0\leq l\leq n(p-1)}\{\sum\limits_{q=0}^{l(p)}(-1)^q\cdot C_n^q\cdot C_{n+l-qp-1}^{l-qp}\}+\mathrm{rk}(\mathscr{E})-2}{p}\cdot\max\{0,\ L_{\max}(\Omega_X^1)\} \right.\\
& +\min\{\ l,\ [\frac{n}{2}](p-1)\ \}\cdot(\frac{n-1}{p}\cdot\max\{0,\ L_{\max}(\Omega_X^1)\}+\mathrm{I}(\Omega_X^1))\\
& \left. +\mathrm{I}(\mathscr{E}) \right\}\cdot p^{n-1}\cdot\mathrm{rk}(\mathscr{E})
\end{aligned}$$

where $l(p)\geq 0$ is the unique integer such that $0\leq l-l(p)<p$.

Proof. For any integer $0 \leq l \leq n(p-1)$, by [14, Lemma 4.3], Theorem 3.5 and Proposition 4.2, we have

$$\begin{aligned}
\mathrm{I}(\mathscr{E}\otimes_{\mathscr{O}_X}\mathrm{T}^l(\Omega_X^1)) \quad \leq \quad & \frac{\mathrm{rk}(\mathrm{T}^l(\Omega_X^1))+\mathrm{rk}(\mathscr{E})-2}{p}\cdot\max\{0,\ L_{\max}(\Omega_X^1)\}\\
& +\mathrm{I}(\mathrm{T}^l(\Omega_X^1))+\mathrm{I}(\mathscr{E})\\
\leq \quad & \frac{\sum\limits_{q=0}^{l(p)}(-1)^q\cdot C_n^q\cdot C_{n+l-qp-1}^{l-qp}+\mathrm{rk}(\mathscr{E})-2}{p}\cdot\max\{0,\ L_{\max}(\Omega_X^1)\}\\
& +\min\{\ l,\ [\frac{n}{2}](p-1)\ \}\cdot(\frac{n-1}{p}\cdot\max\{0,\ L_{\max}(\Omega_X^1)\}+\mathrm{I}(\Omega_X^1))\\
& +\mathrm{I}(\mathscr{E})\\
& \text{where } l(p)\geq 0 \text{ is the unique integer such that } 0\leq l-l(p)<p.
\end{aligned}$$

where the first inequality is followed by Proposition 4.2, and the second inequality is followed by Theorem 3.5. Submit the above inequality into inequality (2), we get the upper bound of $\mathrm{I}(F_{X*}(\mathscr{E}))$ as expected. □

We give some sufficient conditions for slope semi-stability of Frobenius direct images of torsion free sheaves as following.

Proposition 4.4. *If Ω_X^1 is a slope strongly semi-stable sheaf with $\mu(\Omega_X^1) \geq 0$. Then $F_{X*}(\mathscr{E})$ is slope semi-stable whenever $\mathscr{E}$ is slope strongly semi-stable. Moreover, if Ω_X^1 is a slope semi-stable sheaf with $\mu(\Omega_X^1) = 0$, then $F_{X*}(\mathscr{E})$ is slope strongly semi-stable whenever $\mathscr{E}$ is slope semi-stable.*

Proof. By Proposition 3.6, we have $\mathrm{T}^l(\Omega_X^1)(0 \leq l \leq \mathrm{rk}(\mathscr{E})(p-1))$ are slope strongly semi-stable. Then $\mathscr{E}\otimes_{\mathscr{O}_X}\mathrm{T}^l(\Omega_X^1)(0 \leq l \leq n(p-1))$ are also slope strongly semi-stable, since tensor product of slope strongly semi-stable sheaves is also slope strongly semi-stable. This implies the slope semi-stability of $F_{X*}(\mathscr{E})$ by [14, Corollary 4.9]. Moreover, if $\mu(\Omega_X^1) = 0$, then slope semi-stability of sheaves is equivalent to the slope strongly semi-stability. Hence $F_{X*}(\mathscr{E})$ is slope strongly semi-stable whenever $\mathscr{E}$ is slope semi-stable, which is also an immediate corollary of [9, Theorem 2.1] and Theorem 4.3. □

5. Slope (semi)-stability of sheaves of locally exact and closed differential forms

Let

$$\Omega^\bullet_X : 0 \longrightarrow \mathscr{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d_1} \Omega^2_X \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \Omega^n_X \longrightarrow 0$$

be the *de Rham complex* of X. Taking Frobenius push forwards F_{X*} to $\Omega^\bullet_X$, we obtain the following complex $F_{X*}(\Omega^\bullet_{X/S})$ on X:

$$\begin{aligned} 0 \longrightarrow F_{X*}(\mathscr{O}_X) &\xrightarrow{F_{X*}(d)} F_{X*}(\Omega^1_X) \xrightarrow{F_{X*}(d_1)} F_{X*}(\Omega^2_X) \xrightarrow{F_{X*}(d_2)} \\ \cdots &\xrightarrow{F_{X*}(d_{n-2})} F_{X*}(\Omega^{n-1}_X) \xrightarrow{F_{X*}(d_{n-1})} F_{X*}(\Omega^n_X) \longrightarrow 0. \end{aligned}$$

The subsheaves $(0 \leq i \leq n)$

$$B^i_X := \mathrm{Im}(F_{X*}(d_{i-1}) : F_{X*}(\Omega^{i-1}_X) \longrightarrow F_{X*}(\Omega^i_X))$$

$$Z^i_X := \mathrm{Ker}(F_{X*}(d_i) : F_{X*}(\Omega^i_X) \longrightarrow F_{X*}(\Omega^{i+1}_X))$$

of $F_{X*}(\Omega^i_X)$ are called the *sheaf of locally exact i-forms* and *sheaf of locally closed i-forms*. For the sake of convenience, we set $B^0_X := 0$, $Z^0_X := \mathscr{O}_X$, $Z^n_X := F_{X*}(\omega_X)$, where $\omega_X := \Omega^n_X$ is the canonical invertible sheaf of X. By Cartier isomorphism [5, Theorem 7.2], we have isomorphisms $\Omega^i_X \cong Z^i_X/B^i_X (0 \leq i \leq n)$.

For a smooth projective curve X of genus $g \geq 2$. M. Raynaud [11] showed that B^1_X is slope semi-stable and K. Joshi [3] proved that B^1_X is indeed slope stable. For a smooth projective surface X such that Ω^1_X is slope semi-stable with $\mu(\Omega^1_X) > 0$, Y. Kitadai and H. Sumihiro [6] showed that B^1_X and B^2_X are also slope semi-stable. Moreover, X. Sun [15] showed that B^1_X and B^2_X are indeed slope stable, Z^1_X is slope semi-stable if $\mathrm{char}(k) = 3$ and Z^1_X is slope stable if $\mathrm{char}(k) > 3$. In the higher dimensional case, X. Sun [15, Theorem 2.3] showed that when $\mathrm{T}^l(\Omega^1_X)(0 \leq l < n(p-1))$ are slope semi-stable with $\mu(\Omega^1_X) > 0$, then B^1_X is slope stable.

Fix an ample divisor H on X, then for any torsion free sheaf $\mathscr{E}$ on X, we have the following formula (cf. [14, Lemma 4.2])

$$\mu(F_{X*}(\mathscr{E})) = \frac{1}{p}\mu(F^*_X F_{X*}(\mathscr{E})) = \frac{n \cdot (p-1)}{2p} \cdot \mu(\Omega^1_X) + \frac{\mu(\mathscr{E})}{p}.$$

Using induction on m, it is easy to induce the following formula

$$\mu(F^m_{X*}(\mathscr{E})) = \frac{n \cdot (p^m-1)}{2p^m} \cdot \mu(\Omega^1_X) + \frac{\mu(\mathscr{E})}{p^m}.$$

It follows that

$$\deg(F_{X*}(\Omega^i_X)) = \frac{n \cdot C^i_n p^{n-1}(p-1)}{2} \cdot \mu(\Omega^1_X) + i \cdot C^i_n p^{n-1} \cdot \mu(\Omega^1_X).$$

Lemma 5.1. *For any integer $0 \leq i \leq n$, we have*

$$\mathrm{rk}(B^i_X) = C^{i-1}_{n-1}(p^n-1), \ \mathrm{rk}(Z^i_X) = C^{i-1}_{n-1}(p^n-1) + C^i_n$$

$$\deg(B^i_X) = \frac{n \cdot C^{i-1}_{n-1} p^{n-1}(p-1)}{2} \cdot \mu(\Omega^1_X) + \sum_{j=1}^{i-1}(-1)^{i+j+1} j \cdot C^j_n(p^{n-1}-1) \cdot \mu(\Omega^1_X)$$

$$\deg(Z^i_X) = \frac{n \cdot C^{i-1}_{n-1} p^{n-1}(p-1)}{2} \cdot \mu(\Omega^1_X) + \sum_{j=1}^{i-1}(-1)^{i+j+1} j \cdot C^j_n(p^{n-1}-1) \cdot \mu(\Omega^1_X) + i \cdot C^i_n \cdot \mu(\Omega^1_X).$$

Proof. Using induction on i. When $i = 0$, there is nothing to prove. Suppose the Lemma is true for $i-1$. Consider the exact sequence

$$0 \to Z_X^{i-1} \to F_{X*}(\Omega_X^{i-1}) \to B_X^i \to 0.$$

Then we have $\mathrm{rk}(B_X^i) = \mathrm{rk}(F_{X*}\Omega_X^{i-1}) - \mathrm{rk}(Z_X^{i-1}) = C_{n-1}^{i-1}(p^n-1)$. and

$$\deg(B_X^i) = \frac{n \cdot C_{n-1}^{i-1}p^{n-1}(p-1)}{2} \cdot \mu(\Omega_X^1) + \sum_{j=1}^{i-1}(-1)^{i+j+1} j \cdot C_n^j(p^{n-1}-1) \cdot \mu(\Omega_X^1).$$

On the other hand, by exact sequence $0 \to B_X^i \to Z_X^i \to \Omega_X^i \to 0$, we have $\mathrm{rk}(Z_X^i) = \mathrm{rk}(B_X^i) + \mathrm{rk}(\Omega_X^i) = C_{n-1}^{i-1}(p^n-1) + C_n^i$, and

$$\deg(Z_X^i) = \frac{n \cdot C_{n-1}^{i-1}p^{n-1}(p-1)}{2} \cdot \mu(\Omega_X^1) + \sum_{j=1}^{i-1}(-1)^{i+j+1} j \cdot C_n^j(p^{n-1}-1) \cdot \mu(\Omega_X^1) + i \cdot C_n^i \cdot \mu(\Omega_X^1).$$

This completes the proof of the Lemma. □

Proposition 5.2. *If $\mu(\Omega_X^1) > 0$. Then*

(1). *$Z_X^i (1 \leq i < \frac{n}{2})$ are never slope semi-stable.*

(2). *If $n \geq 3$ and $\mathrm{T}^l(\Omega_X^1)(0 \leq l < n(p-1))$ are slope semi-stable. Then the Harder-Narasimhan filtration of Z_X^1 is*

$$0 \subset B_X^1 \subset Z_X^1.$$

Proof. (1). Consider the exact sequence $0 \to B_X^i \to Z_X^i \to \Omega_X^i \to 0$. To prove Z_X^i is not slope semi-stable, it is enough to prove $\mu(B_X^i) > \mu(\Omega_X^i)$, i.e.

$$\frac{n \cdot C_{n-1}^{i-1}p^{n-1}(p-1) + 2\sum_{j=1}^{i-1}(-1)^{i+j+1} j C_n^j(p^{n-1}-1)}{2C_{n-1}^{i-1}(p^n-1)} > i.$$

Since $\sum_{j=1}^{i-1}(-1)^{i+j+1} j \cdot C_n^j(p^{n-1}-1) > 0$, so we only have to show that

$$\frac{n \cdot C_{n-1}^{i-1}p^{n-1}(p-1)}{2C_{n-1}^{i-1}(p^n-1)} = \frac{n(p^n-p)}{2(p^n-1)} > i.$$

The above inequality is trivial when $n > 2 \cdot i \geq 2$.

(2). Since B_X^1 is slope semi-stable(cf. [15, Theorem 2.3]), and by Lemma 5.1 we have $\mu(B_X^1) > \mu(\Omega_X^1)$ if $n \geq 3$. Then by exact sequence

$$0 \to B_X^1 \to Z_X^1 \to \Omega_X^1 \to 0,$$

we know the Harder-Narasimhan filtration of Z_X^1 is $0 \subset B_X^1 \subset Z_X^1$. □

Proposition 5.3. *If Ω_X^1 is slope semi-stable with $\mu(\Omega_X^1) = 0$. Then $B_X^i (1 \leq i \leq n)$ and $Z_X^i (1 \leq i \leq n-1)$ are slope strongly semi-stable.*

Proof. By [9, Theorem 2.1], we have Ω_X^1 is slope strongly semi-stable, hence Ω_X^i is slope strongly semi-stable for any integer $1 \leq i \leq n$. Then by Proposition 4.4, $F_{X*}(\Omega_X^i)(1 \leq i \leq n)$ are slope strongly semi-stable with $\mu(F_{X*}(\Omega_X^i)) = 0$. By Lemma 5.1 we have $\mu(B_X^i) = \mu(Z_X^i) = 0$, $1 \leq i \leq n$. Then the Proposition follows from the exact sequence $0 \to Z_X^{i-1} \to F_{X*}(\Omega_X^{i-1}) \to B_X^i \to 0$. □

Proposition 5.4. *If $\mu(\Omega^1_X) > 0$ and $\mathrm{T}^l(\Omega^1_X)(0 \leq l \leq n(p-1))$ are slope semi stable. Then for any subsheaf $B \subset F_{X*}\omega_X$ with $0 < \mathrm{rk}(B) < \mathrm{rk}(F_{X*}\omega_X)$, we have*

$$\mu(B) - \mu(B^n_X) \leq -\frac{n(p-1)(p^n - \mathrm{rk}(B) - 1)}{2p(p^n-1)\cdot \mathrm{rk}(B)}\mu(\Omega^1_X).$$

In particular, B^n_X is slope stable.

Proof. Since $0 < \mathrm{rk}(B) < \mathrm{rk}(F_{X*}\omega_X)$, by Theorem [15, Theorem 2.3], we have

$$\mu(B) \leq \mu(F_{X*}\omega_X) - \frac{n(p-1)}{2p\cdot \mathrm{rk}(B)}\cdot \mu(\Omega^1_X) = \frac{n(p+1)\cdot \mathrm{rk}(B) - n(p-1)}{2p\cdot \mathrm{rk}(B)}\cdot \mu(\Omega^1_X).$$

By Lemma 5.1 and the combinational formula $\sum_{j=0}^{n}(-1)^j j \cdot C^j_n = 0$, we have

$$\mu(B) - \mu(B^n_X) \leq -\frac{n(p-1)(p^n - \mathrm{rk}(B) - 1)}{2p(p^n-1)\cdot \mathrm{rk}(B)}\cdot \mu(\Omega^1_X).$$

Let sub-sheaf $B \subseteq B^n_X$ with $0 < \mathrm{rk}(B) < \mathrm{rk}(B^n_X) = p^n - 1$. Then $\mu(B) < \mu(B^n_X)$. Hence, B^n_X is slope stable. □

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